

HYPERKÄHLER METRICS BUILDING AND INTEGRABLE MODELS

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ABSTRACT

Methods developed for the analysis of integrable systems are used to study the problem of hyperKähler metrics building as formulated in $D = 2$ $N = 4$ supersymmetric harmonic superspace. We show, in particular, that the constraint equation $\beta \partial^{++2} \omega - \xi^{++2} \exp 2\beta \omega = 0$ and its Toda like generalizations are integrable. Explicit solutions together with the conserved currents generating the symmetry responsible of the integrability of these equations are given. Other features are also discussed.

1 Introduction

Recently much interest has been shown in the study of nonlinear integrable models from different points of view [1, 2]. These are exactly solvable systems exhibiting infinite dimensional local symmetries and are involved in various areas of mathematical physics. The aim of this paper is to draw the basic lines of a new area where the development made in integrable theories can be used. This area deals with the construction of new explicit hyperKähler metrics of the four dimensional euclidean gravity and their generalizations. Recall that the problem of hyperKähler metrics building is an interesting question of hyperKähler geometry that can be solved in a nice way in harmonic superspace HS [3, 4] if one knows how to solve the following nonlinear differential equations on the sphere S^2 :

$$\begin{aligned} \partial^{++} q^+ - \partial^{++} \left(\frac{\partial V^{4+}}{\partial(\partial^{++} \bar{q}^+)} \right) + \frac{\partial V^{4+}}{\partial \bar{q}^+} &= 0 \\ \partial^{++} \bar{q}^{++} + \partial^{++} \left(\frac{\partial V^{4+}}{\partial(\partial^{++} \bar{q}^+)} \right) - \frac{\partial V^{4+}}{\partial q^+} &= 0 \end{aligned} \tag{1}$$

where $q^+ = q^+(z, \bar{z}, u^\pm)$ and its conjugates $\bar{q}^+ = q^+(z, \bar{z}, u^\pm)$ are complex fields defined on $R^2 \otimes S^2 \approx \mathcal{C} \otimes S^2$, respectively, parametrized by the local analytic coordinates z, \bar{z} and the harmonic variables u^\pm . $\partial^{++} = u^{+i} \partial / \partial u^{-i}$ is the so-called harmonic derivative. $V^{4+} = V^{4+}(q, u)$ is an interacting potential depending in general on q^+, \bar{q}^+ , their derivatives and the u^\pm 's. As described in [3], the fields q^+ and \bar{q}^+ are globally defined on the sphere $S^2 \approx SU(2)/U(1)$ and may be expanded into an infinite series preserving the total charge, as shown here below:

$$q^+(z, \bar{z}, u) = u_i^+ \varphi^i(z, \bar{z}) + u_i^+ u_j^+ u_k^- \varphi^{(ijk)}(z, \bar{z}) + \dots \tag{2}$$

Note that Eqs.(1), which fix the u -dependence of the q^+ 's, is in fact the pure bosonic projection of a two dimensional $N = 4$ supersymmetric HS superfield equation of motion [4]. The remaining equations carry the spinor contributions. They describe among other things the space time dynamics of the physical degrees of freedom, namely the four bosons $\varphi^i(z, \bar{z})$; $i = 1, 2$ and their $D = 2$ $N = 4$ supersymmetric partners. For more details see Section 2.

An equivalent way of writing Eqs.(1) is to use the Howe–Stelle–Townsend (HST) realization of the $D = 2$ $N = 4$ hypermultiplet $(0^4, (1/2)^4)$ [5, 3]. In this representation, that will be used in this paper, Eqs.(1) read as

$$\partial^{++2} \omega - \partial^{++} \left[\frac{\partial H^{4+}}{\partial(\partial^{++} \omega)} \right] + \frac{\partial H^{4+}}{\partial \omega} = 0, \tag{3}$$

where $\omega = \omega(z, \bar{z}, u)$ is a real field defined on $\mathcal{C} \otimes S^2$ and whose leading terms of its harmonic expansion read as:

$$\omega(z, \bar{z}, u) = u_i^+ u_j^- f^{ij}(z, \bar{z}) + u_i^+ u_j^+ u_k^- u_\ell^- g^{ijkl}(z, \bar{z}) + \dots \tag{4}$$

Similary as for Eqs.(1), the interacting potential H^{4+} depends in general on ω , its derivatives and on the harmonics. In the remarkable case where the potentials V^{4+} and H^{4+}

do not depend on the derivatives of the fields q^+ and ω , Eqs.(1) and (3) reduce to

$$\partial^{++} q^+ + \frac{\partial V^{4+}}{\partial q^+} = 0 \quad (5)$$

$$\partial^{++2} \omega + \frac{\partial H^{4+}}{\partial \omega} = 0 . \quad (6)$$

Remark that the solutions of these equations depend naturally on the potentials V^{4+} and H^{4+} and then the finding of these solutions is not an easy question. There are only few examples that had been solved exactly. The first example we give is the Taub–Nut model leading to the well-known Taub–Nut metric of the four dimensional euclidean gravity. Its potential $V^{4+}(q^+, \bar{q}^+)$ is given by [4]

$$V^{4+} = \frac{\lambda}{2} (q^+ \bar{q}^+)^2 , \quad (7)$$

where λ is a real coupling constant. Putting Eq.(7) back into Eq.(5), one gets

$$\partial^{++} q^+ + \lambda(q^+ \bar{q}^+) \bar{q}^+ = 0 \quad (8)$$

whose solution reads as [4]

$$q^+(z, \bar{z}, u) = u_i^+ \varphi^i(z, \bar{z}) \exp -\lambda(u_k^+ u_\ell^- \varphi^k \bar{\varphi}^\ell) . \quad (9)$$

Following Ref.[4], the knowledge of the solution Eq.(9) of Eq.(8) is the key point in the identification of the metric of the manifold parametrized by the bosonic fields $\varphi^i(z, \bar{z})$ and $\bar{\varphi}^i(z, \bar{z})$ of the $D = 2$ $N = 4$ supersymmetric nonlinear Taub–Nut σ -model whose bosonic part reads as [4]

$$S_B^{TN} = -\frac{1}{2} \int dz d\bar{z} \left(g_{ij} \partial_z \varphi^i \partial_{\bar{z}} \varphi^j + \bar{g}^{ij} \partial_z \bar{\varphi}_i \partial_{\bar{z}} \bar{\varphi}_j + 2h_j^i \partial_z \varphi^j \partial_{\bar{z}} \bar{\varphi}_i \right)$$

where

$$\begin{aligned} g_{ij} &= \frac{\lambda(2 + \lambda f \bar{f})}{2(1 + \lambda f \bar{f})} \bar{f}_i \bar{f}_j, \quad \bar{g}^{ij} = \frac{\lambda(2 + f \bar{f})}{2(1 + \lambda f \bar{f})} f^i f^j \\ h_j^i &= \delta_j^i (1 + \lambda f \bar{f}) - \frac{\lambda(2 + \lambda f \bar{f})}{2(1 + \lambda f \bar{f})} f^i \bar{f}_j, \\ f \bar{f} &= f^i \bar{f}_i . \end{aligned} \quad (10)$$

Moreover, using the HST representation of the $D = 2$ $N = 4$ hypermultiplet, Eq.(8) may be rewritten as

$$\left[\partial^{++} + \lambda \frac{\omega \overleftrightarrow{\partial}^{++} \bar{\omega}}{(1 + 2\omega \bar{\omega})} \right]^2 \cdot \omega = 0 , \quad (11)$$

where $\bar{\omega}$ is the complex conjugate of ω and $\omega \overleftrightarrow{\partial}^{++} \bar{\omega} = \omega \partial \bar{\omega} - \partial \omega \cdot \bar{\omega}$. Here also, this equation is exactly solvable. The solution reads as

$$\omega(z, \bar{z}, u) = u_i^+ u_j^- f^{ij}(z, \bar{z}) \exp -i\lambda\beta \quad (12)$$

where

$$\beta = u_k^+ u_\ell^- \left[f^0 \bar{f}^{(k\ell)} - \bar{f}^0 f^{(ij)} + \epsilon_{rs} f^{(ks)} \bar{f}^{(\ell r)} \right] . \quad (13)$$

The explicit form of the Taub–Nut metric in the ω –representation is worked out in Ref.[6].

The second example that has been solved exactly is the Eguchi–Hanson model whose potential reads as [7]

$$H^{4+}(\omega) = \left[u_i^+ u_j^+ \xi^{(ij)} \right]^2 / \omega^2 , \quad (14)$$

where ξ^{ij} is an $SU(2)$ real constant triplet. Details are exposed in [7]. It is interesting to note here that for the above mentioned nonlinear differential equations and their generalizations, the integrability is due to the existence of symmetries allowing their linearizations.

In this paper, we focus our attention on Eq.(6) and look for potentials leading to exact solutions of this equation. Our method is based on suggesting new plausible integrable equations by proceeding by formal analogy with the known integrable two dimensional nonlinear differential equations especially the Liouville equation and its Toda generalizations.

Among our results we show that the model described by the potential

$$H^{4+}(\omega, u) = -\frac{1}{2} \left(\frac{\xi^{++}}{\lambda} \right)^2 \exp 2\lambda\omega \quad (15)$$

which implies in turn the following equation

$$\lambda \partial^{++2}\omega - \xi^{++2} \exp 2\lambda\omega = 0 \quad (16)$$

is integrable. The explicit solution of this nonlinear differential equation reads as

$$\xi^{++} \exp \lambda\omega = \frac{u_i^+ u_j^+ f^{ij}(z, \bar{z})}{1 - u_k^+ u_\ell^- f^{k\ell}(z, \bar{z})} . \quad (17)$$

Here also we show that the integrability of Eq.(16) is due to the existence of a symmetry generated by the following conserved current

$$t^{4+} = (\partial^{++}\omega)^2 - \frac{1}{\lambda} \partial^{++2}\omega \quad (18)$$

$$\partial^{++} t^{4+} = 0 .$$

This representation of the current t^{4+} , which in some sense resembles to the Liouville current, can also be obtained by using the field theoretical method or again with the help of an extended Miura transformation which reads, in the general situation, as

$$(\partial^{++n} - W^{2n+}) = \prod_{j=1}^n (\partial^{++} - V_j^{++}) , \quad (19)$$

where the fields $V_j^{++}, j = 1, \dots, n$, obeys the traceless condition namely $\sum_{j=1}^n V_j^{++} = 0$.

The extension of Eqs.(15)–(16) and (18) for a rank n simple Lie algebra is also studied.

The presentation of this paper is as follows: First we review briefly the hyperKähler metrics building from the HS method and show that the solving of the pure bosonic Eq.(6) is the main step to achieve in this programme. Then, we present our integrable model and discuss its Toda like generalizations. After that we introduce the generalized Miura transformation and derive a series of conserved currents $W^{2n+}, n \geq 1$ responsible for the integrability of the generalizations of our model. Finally, we give our conclusion.

2 Generalities on the hyperKähler metrics building from HS

HyperKähler metrics have vanishing Ricci tensors and are then natural solutions of the Einstein equation in the vacuum [8]. They also appear as the metrics of the bosonic manifolds of two dimensional $N = 4$ supersymmetric nonlinear σ -models describing the self coupling of the hypermultiplet $(0^4, (1/2)^4)$ [9, 4, 5, 7]. A powerful method to write down the general form of such models, and then build the underlying hyperKähler metrics, is given by harmonic superspace. The latter is parametrized by the supercoordinates $Z^M = (z_A^M, \theta_r^-, \bar{\theta}_r^-)$ where $z_A^M = (z, \bar{z}, \theta_r^+, \bar{\theta}_r^+, u^\pm)$ are the supercoordinates of the so-called analytic subspace in which $D = 2$ $N = 4$ supersymmetric theories are formulated [3]. $d^2 z d^4 \theta^+ du$ is the HS integral measure in the z_A^M basis. The matter superfield $(0^4, (1/2)^4)$ is realized by two dual analytic superfields $Q^+ = Q^+(z, \bar{z}, \theta^+, \bar{\theta}^+, u)$ and $\Omega = \Omega(z, \bar{z}, \theta^+, \bar{\theta}^+, u)$ whose leading bosonic fields are respectively given by q^+ and ω Eqs.(2) and (4).

The action describing the general coupling of the analytic superfield Ω we are interested in here, reads as [3]:

$$S[\Omega] = \int d^2 z d^4 \theta^+ du \left(\frac{1}{2} (D^{++}\Omega)^2 - H^{4+}(\Omega, u) \right), \quad (20)$$

where the harmonic derivative D^{++} is given by

$$D^{++} = \partial^{++} - 2\bar{\theta}_r^+ \theta_r^+ \partial_{-2r}. \quad (21)$$

The superfield equation of motion

$$D^{++2}\Omega - D^{++} \left(\frac{\partial H^{4+}}{\partial D^{++}\Omega} \right) + \frac{\partial H^{4+}}{\partial \Omega} = 0 \quad (22)$$

which depends naturally on the interaction, contains as many differential equations as the component fields carried by the superfield Ω . It turns out that this system of differential equations splits into three subsets of equations depending on the canonical dimensions of the component fields of the superfield Ω . To be more precise let us describe briefly the main steps of the hyperKähler metrics building procedure.

1. Specify the self interacting potential $H^{4+}(\Omega, u)$, since to each interaction corresponds a definite metric. As an example, one may take the Taub–Nut interaction $\lambda(Q^+ \bar{Q}^+)^2$ [4], or again the Eguchi–Hanson one: $(\xi^{++}/\Omega)^2$ [7]. These potentials do not depend on the superfield derivatives and lead to equations type (5) and (6). Another interesting example that will be considered in this study is given by:

$$H^{4+}(\Omega, u^+) = -\frac{1}{2} (\xi^{++}/\beta)^2 \exp 2\beta\Omega, \quad (23)$$

where β is a coupling constant and $\xi^{++} = u_i^+ u_j^+ \xi^{(ij)}$, a constant isotriplet similar to that appearing in the Eguchi–Hanson model.

2. Write down the corresponding superfield equation of motion, namely

$$D^{++2} \Omega + \frac{\partial H^{4+}}{\partial \Omega} = 0, \quad (24)$$

which reads, for the potential (23), as:

$$\beta D^{++2} \Omega - (\xi^{++})^2 \exp 2\beta\Omega = 0 . \quad (25)$$

3. Expanding the analytic superfield Ω in θ_r^+ and $\bar{\theta}_r^+$ series as

$$\begin{aligned} \Omega = & \omega + (\theta_-^+ \theta_+^+ F^{--} + \bar{\theta}_+^+ \bar{\theta}_-^+ \bar{F}^{--}) + (\bar{\theta}_-^+ \theta_+^+ G^{--} + \\ & \bar{\theta}_+^+ \theta_-^+ \bar{G}^{--} + (\bar{\theta}_-^+ \theta_-^+ B_{++}^{--} + \bar{\theta}_+^+ \theta_+^+ B_{--}^{--} + \\ & \bar{\theta}_-^+ \theta_-^+ \bar{\theta}_+^+ \theta_+^+ \Delta^{(-4)}) , \end{aligned} \quad (26)$$

where we have set the spinor fields to zero for simplicity. Then putting back into Eq.(24) one obtains three kinds of differential equations: The first one which is given by the equation of motion of the Lagrange field $\Delta^{(-4)}$ of canonical dimension 2, reads as:

$$\partial^{++2} \omega + \frac{\partial H^{4+}(\omega)}{\partial \omega} = 0 . \quad (27)$$

This is a constraint equation fixing the dependence of ω in terms of the free bosonic fields f^{ij} of the $D = 2$ $N = 4$ hypermultiplet. The knowledge of the solution of this nonlinear equation is necessary as it is one of the two main difficult steps in the construction of hyperKähler metrics in this way. In the next section, we shall show that the methods developed in integrable theories can be used to solve a specific class of these equations such that the equation implied by the potential Eq.(23)

$$\beta \partial^{++2} \omega - \xi^{++2} \exp 2\beta\omega = 0 \quad (28)$$

or again its generalizations. The second set of relations are given by the equations of motion of the auxiliary fields F^{--} , G^{--} , and B_{rr}^{--} of canonical dimensions one. In all known cases, the solutions of these equations are obtained by making an appropriate change of variables inspired from the solution of Eq.(27). Note that for the potential (25), the auxiliary fields equations read as

$$\begin{aligned} \beta \partial^{++2} F^{--} - \xi^{++2} F^{--} \exp 2\beta\omega &= 0 \\ \beta \partial^{++2} G^{--} - \xi^{++2} G^{--} \exp 2\beta\omega &= 0 \\ \beta \partial^{++2} B_{rr}^{--} - \xi^{++2} B_{rr}^{--} \exp 2\beta\omega &= 4\partial^{++} \partial_{rr} \omega . \end{aligned} \quad (29)$$

These equations, when solved, give the relation between the auxiliary fields F^{--} , G^{--} , and B_{rr}^{--} and the $D = 2$ matter field ω and its space-time derivatives. Note also that Eqs.(27)–(29) once integrated fix $D = 2$ $N = 4$ supersymmetry partially on shell. The last equation is given by the space-time equation of motion of ω . It describes the dynamics of ω and is not involved in the hyperKähler metric building programme.

4. Combining the results of the steps one, two and three, one finds that the bosonic part of the action Eq.(20) takes a form similar to Eq.(10) from which one can read the hyperKähler metric directly.

At the end of this section we would like to point out that, in general, the energy momentum tensor of the action (20) reads in terms of the real superfield Ω and the interacting potential as:

$$T^{4+}(\Omega) = \frac{1}{2}(D^{++}\Omega)^2 - \frac{\partial H^{4+}}{\partial D^{++}\Omega} \cdot D^{++}\Omega - H^{4+} . \quad (30)$$

The conservation law of this current can easily be checked with the help of the equation of motion (22). For the interacting potential given by Eq.(23), the above conserved current takes the remarkable form

$$T^{4+}(\Omega) = \frac{1}{2}(D^{++}\Omega)^2 - \frac{1}{\beta} D^{++2}\Omega . \quad (31)$$

3 The integrability of $\beta\partial^{++2}\omega - \xi^{++2}\exp 2\beta\omega = 0$

In this section we prove that this nonlinear harmonic differential equation is solvable though apparently it shows no special symmetry. The property of solvability of this equation is expected from its formal analogy with the well-known Liouville equation. This why we shall start by describing the local equation of motion of the two dimensional Liouville field $\varphi(z, \bar{z})$, namely

$$\beta \partial_z \partial_{\bar{z}} \varphi - \exp(2\beta\varphi) = 0 \quad (32)$$

where β is a real coupling constant. To study the integrability of this equation, different techniques including the Lax method were developed. Here, we content ourselves to recall that the explicit solution of this nonlinear equation can be written as [10]:

$$\exp 2\beta\varphi = cte \cdot \frac{F'(z) \cdot \bar{F}'(\bar{z})}{(1 - F(z) \cdot \bar{F}(\bar{z}))^2} , \quad (33)$$

where $F(z)$ and $\bar{F}(\bar{z})$ are arbitrary analytic and anti-analytic functions, $F'(z) = \partial F(z)$ and $\bar{F}'(\bar{z}) = \partial \bar{F}(\bar{z})$. As it is well known, the integrability of the nonlinear differential Liouville equation is due to its conformal symmetry generated by the following classical energy momentum tensor

$$T_L(\varphi) = (\partial\varphi)^2 - \frac{1}{\beta}(\partial^2\varphi) . \quad (34)$$

The conservation law of this current follows immediately by using the equation of motion as shown here below.

$$\begin{aligned} \bar{\partial} T_L(\varphi) &= 2\partial\varphi\partial\bar{\varphi} - \frac{1}{\beta} \partial(\partial\varphi) \\ &= -\frac{1}{\beta}(\partial - 2\beta\partial\varphi) \partial\varphi = 0 . \end{aligned} \quad (35)$$

Having given the necessary ingredient of the classical Liouville equation, we pass now to study our equation.

$$\beta \partial^{++2}\omega - \xi^{++2}\exp 2\beta\omega = 0 . \quad (36)$$

This is a nonlinear harmonic differential equation which in principle, is not easy to solve. However, forgetting about the constant ξ^{++} and the global properties of the field ω with respect to the coordinates of S^2 , Eq.(16) shows a striking resemblance with the Liouville equation examined earlier. Therefore, one should expect that both these equations would share some general features and more particularly their integrabilities. Using this formal analogy with the Liouville equation, it is not difficult to see that the solution of Eq.(16)

can be expressed in terms of the four physical real bosonic fields f^{ij} of the $D = 2$ $N = 4$ HST free hypermultiplet $(0^4, (1/2)^4)$ as follows:

$$\xi^{++} \exp \beta \omega = \frac{u_i^+ u_j^+ f^{ij}}{1 - u_i^+ u_j^- f^{ij}} , \quad (37)$$

To check that Eq.(37) is indeed the solution of Eq.(16), note first of all that we have

$$\beta \partial^{++} \omega = \frac{u_i^+ u_j^+ f^{ij}}{1 - u_k^+ u_\ell^- f^{k\ell}} , \quad (38)$$

which with the help of Eq.(37), it also reads as:

$$\beta \partial^{++} \omega = \xi^{++} \exp \beta \omega . \quad (39)$$

Acting on this relation by the harmonic differential operator ∂^{++} , we get after setting $f^{++} = u_i^+ u_j^+ f^{ij}$ and $f = u_i^+ u_j^- f^{ij}$ for simplicity:

$$\beta \partial^{++2} \omega = \left(\frac{f^{++}}{1 - f} \right)^2 . \quad (40)$$

Using Eq.(37) once again, one sees that Eq.(40) can be rewritten as

$$\beta \partial^{++2} \omega = \xi^{++2} \exp 2\beta \omega , \quad (41)$$

which coincides exactly with Eq.(16). Note by the way that the solution (37) looks, in some sense, like that given by Eq.(33). Thus the question is, can we find appropriate symmetry responsible of the integrability of Eq.(16)? The answer to this question is obtained by exploiting once more the analogy with the symmetry of the Liouville equation (32). There, the symmetry is the conformal invariance generated by the conserved current Eq.(34). In our case the classical current generating the symmetry of Eq.(16) is expected to have the following natural form:

$$t^{4+}(\omega) = (\partial^{++} \omega)^2 - 1/\beta \partial^{++2} \omega \quad (42)$$

which is just the pure bosonic projection of Eq.(31). Here also the conservation law of this current follows by using Eq.(16). Indeed we have

$$\begin{aligned} \partial^{++} t^{4+} &= 2(\partial^{++} \omega)(\partial^{++2} \omega) - 1/\beta \partial^{++3} \omega \\ &= -1/\beta^2 (\partial^{++} - 2\beta \partial^{++} \omega) \beta \partial^{++2} \omega , \end{aligned} \quad (43)$$

which vanishes identically with the help of the identity (41).

Furthermore, knowing that integrable two dimensional field theoretical models are intimately related to the simple roots system of Lie algebras, our goal in the next discussion is to use this crucial property to test our expected integrable model Eq.(40). The Liouville equation discussed at the beginning of this section is in fact the leading case of a system of integrable equations describing the so-called A_n -Toda models with $n = 1, 2, \dots$. Denoting by $\{\vec{\alpha}_i; 1 \leq i \leq n\}$ the simple roots system of the A_n -Lie algebra and by

$$\vec{\omega} = \sum_{i=1}^n \vec{\alpha}_i \omega_i , \quad (44)$$

an $O(n)$ vector of n component fields ω_i , the A_n -Toda like extension of Eq.(36) reads as:

$$\beta \partial^{++2} \vec{\omega} - \xi^{++2} \sum_{j=1}^n \vec{\alpha}_j \exp \beta \vec{\alpha}_j \cdot \vec{\omega} = 0 . \quad (45)$$

The integrability of these equations is ensured by showing the existence of n independent conserved currents type Eq.(42) generating their underlying symmetries. This will be done in the next section. We end this study by noting that Eq.(45) is just the pure bosonic projection of a superfield equation of motion

$$\beta D^{++2} \Omega - \xi^{++2} \sum_{j=1}^n \vec{\alpha}_j \exp \beta \vec{\alpha}_j \cdot \vec{\Omega} , \quad (46)$$

obtained by varying the following HS superspace action

$$S[\Omega] = \int d^2z d^2\theta^+ \left\{ \frac{1}{2} D^{++} \vec{\Omega} \cdot D \vec{\Omega} - (\xi^{++}/\beta)^2 \sum_{i=1}^n \exp \beta \vec{\alpha}_i \vec{\Omega} \right\} . \quad (47)$$

4 Generalized Miura transformation

We start by recalling that for bosonic Toda conformal field theories based on the simple Lie algebra A_n , the field realization of the higher spin currents is given by the so-called WA_n Miura transformation namely [11]:

$$\partial_z^n - \sum_{k=2}^n u_k \partial^{n-k} = \prod_{j=1}^n (\partial_z - q_{zj}) , \quad (48)$$

where the q_{zj} 's, $j = 1, \dots, n$, are spin one analytic fields obeying $\sum_{j=1}^n q_{zi} = 0$ and where we have used the convention notation:

$$(\partial_z - q_z)_j = (\partial_z - q_{zj}) , \quad (49)$$

Expanding the r.h.s of Eq.(48), one gets the field realization of the conformal spin k conserved currents u_k . Naturally this method applies for super TCFT's as well [11]. In our present case, one can define an adapted Miura transformation by generalizing the analysis of the beginning of Section 3 based on the formal analogy between the Liouville theory and Eq.(16). This transformation, to which we shall refer hereafter to as the generalized Miura transformation, reads in the language of HS superfields as:

$$(D^{++})^n - \sum_{k=2}^n J^{+2k} (D^{++})^{n-k} = \prod_{j=1}^n (D^{++} - V^{++})_j , \quad (50)$$

where

$$(D^{++} - V^{++})_j = D^{++} - V_j^{++} \quad (51)$$

$$\sum_{j=1}^n V_j^{++} = 0 .$$

Expanding the r.h.s of Eq.(50), one naturally gets the superfield realization of the conserved current J^{+2k} . Let us describe briefly hereafter the $n = 2$ and 3 situations. In the first case, Eq.(50) leads to the following realization of the unique current J^{4+}

$$J^{4+} = (V^{++})^2 - D^{++} V^{++} \quad (52)$$

where we have used $V^{++} = V_2^{++} = -V_1^{++}$. Taking $V^{++} = \beta D^{++} \Omega$ and $J^{++} = \beta^2 T^{4+}$, the above equation reduces to Eq.(31) giving the energy momentum tensor whose pure bosonic projection coincides with Eq.(42). For the $n = 3$ case, the generalized Miura transformation leads to the following superfield realizations of the two supercurrents J^{4+} and J^{6+} :

$$\begin{aligned} J^{4+} &= [(V_1^{++})^2 + 2D^{++}V_1^{++}] + [(V_2^{++})^2 + D^{++}V_2^{++}] + V_1^{++} \cdot V_2^{++} \\ J^{6+} &= -(V_1^{++}V_2^{++})(V_1^{++} + V_2^{++}) + (V_1^{++} + V_2^{++})D^{++}V_1^{++} \\ &\quad - D^{++}(V_1^{++}V_2^{++}) + D^{++2}V_1^{++} \end{aligned} \quad (53)$$

where V_1^{++} and V_2^{++} are two free superfields. Setting $V_i^{++} = \beta D^{++} \Omega_i$, the conservation laws of these currents follows as usual with the help of the equations of motion (46).

5 Conclusion

In this paper, we have shown that the two dimensional integrable model techniques may be used in the accomplishment of the programme of constructing of new hyperKähler metrics. Recall once again that the problem of hyperKähler metrics building can be studied in a convenient way in harmonic superspace. The main difficulty in this approach is the solving of nonlinear harmonic differential equations. Only few and special examples such as Taub–Nut and Eguchi–Hanson models and some of their generalizations were solved exactly in literature. In all these cases, the key point in solving these equations is the existence of a symmetry allowing their linearization. In the present study, using a formal analogy with TCFT's, we have succeeded to draw the main lines of a new class of nonlinear two dimensional $N = 4$ supersymmetric σ -models that can be solved exactly. These models are described by the following HS action

$$S[\Omega] = \int d^2z d^4\theta^+ du \left(\frac{1}{2} D^{++} \vec{\Omega} \cdot D^{++} \vec{\Omega} - (\xi^{++}/\beta)^2 \sum_{i=1}^n \exp \vec{\alpha}_i \vec{\Omega} \right) \quad (54)$$

where $\vec{\alpha}_i, i = 1, \dots, n$, are the simple roots of a rank n simple Lie algebra. $\vec{\Omega} = \sum_{i=1}^n \vec{\alpha}_i \Omega_i$ and $\xi^{++} = u_k^+ u_\ell^+ \xi^{(k\ell)}, k\ell = 1, 2$, is a constant. In the general case we have shown that the pure bosonic projection of the superfield equation of motion, namely

$$\beta \partial^{++2} \vec{\omega} - (\xi^{++})^2 \sum_{i=1}^n \vec{\alpha}_i \exp \beta \vec{\alpha}_i \vec{\omega} = 0 \quad (55)$$

is integrable. The corresponding $(n-1)$ conserved current, responsible of the integrability of the above equation, was obtained by using an adapted Miura transformation. Here it is interesting to note that a Lax formalism similar to that used in TCFT's can be defined also in our case. For the special case $n = 2$, we have also given the explicit solution

of Eq.(16). What remains to do is to write down the explicit form of the underlying hyperKähler metric associated to the potential Eq.(23). This technical problem will be addressed in a future occasion.

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